

## Hardy-Hilbert's integral inequality in new kinds

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**Abstract.** New kinds of Hardy-Hilbert's integral inequalities are presented.

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### 1. Introduction

If  $f, g \geq 0$ ,  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $0 < \int_0^\infty f^p(x)dx < \infty$  and  $0 < \int_0^\infty g^q(x)dx < \infty$ , then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin(\frac{\pi}{p})} \left( \int_0^\infty f^p(x)dx \right)^{1/p} \left( \int_0^\infty g^q(x)dx \right)^{1/q} \quad (1)$$

where the constant factor  $\frac{\pi}{\sin(\frac{\pi}{p})}$  is the best possible. Inequality (1) is called a Hardy-Hilbert's integral inequality (see [1]) and it is important in analysis and applications (cf. Mitrinovic et al.[2]), Hardy et al.[1] gave an inequality similar to (1) as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dx dy \leq pq \left( \int_0^\infty f^p(t)dt \right)^{1/p} \left( \int_0^\infty g^q(t)dt \right)^{1/q} \quad (2)$$

where the constant factor  $pq$  is the best possible.

Other mathematicians present generalizations or new kinds of (2) as follows:

**Theorem 1** (see [3]). *If  $\lambda > 0$ ,  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $f, g \geq 0$  such that*

$$0 < \int_0^\infty t^{p-1-\lambda} f^p(t)dt < \infty \text{ and } 0 < \int_0^\infty t^{q-1-\lambda} g^q(t)dt < \infty,$$

*then one has*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x^\lambda, y^\lambda\}} dx dy \leq \frac{pq}{\lambda} \left( \int_0^\infty t^{p-1-\lambda} f^p(t)dt \right)^{1/p} \left( \int_0^\infty t^{q-1-\lambda} g^q(t)dt \right)^{1/q}, \quad (3)$$

*where the constant factor  $\frac{pq}{\lambda}$  is the best possible.*

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**Theorem 2** (See [4]). Suppose  $f, g \geq 0$ ,  $0 < \int_0^\infty f^2(x)dx < \infty$ ,  $0 < \int_0^\infty g^2(x)dx < \infty$ .

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y+\max\{x,y\}} dx dy \leq c \left( \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right)^{1/2}, \quad (4)$$

where  $c = \sqrt{2}(\pi - 2 \arctan \sqrt{2}) \approx 1.7408$ .

The Gamma and Beta functions denoted by  $\Gamma(p)$ ,  $B(p, q)$ , are defined by

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt, \quad p > 0 \quad (5)$$

and

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} dt, \quad p > 0, \quad q > 0, \quad (6)$$

respectively.

## 2. Main results

**Lemma 1.** Let  $p > 0$ ,  $q > 0$ . Then

$$\Gamma(p) = e \int_0^1 \frac{x^{-p-1} e^{-1/x}}{(1-x)^{1-p}} dx \quad (7)$$

$$\Gamma(p) = e \int_1^\infty \frac{e^{-x}}{(x-1)^{1-p}} dx \quad (8)$$

and

$$B(p, q) = \int_1^\infty \frac{x^{-p-q}}{(x-1)^{1-p}} dx. \quad (9)$$

**Proof.** (7) and (8) follows by putting  $t = x^{-1} - 1$ ,  $x - 1$ , in (5), respectively, and (8) follows by putting  $x = 1/t$  in (6).  $\square$

**Theorem 3.** Assume that  $f, g \geq 0$ ,  $\lambda > 0$ ,  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dx dy \\ & \leq 2 \frac{\Gamma(\lambda)}{e} \left( \int_0^\infty t^{(1-\lambda)(p-1)-1} f^p(t) dt \right)^{1/p} \left( \int_0^\infty t^{(1-\lambda)(q-1)-1} g^q(t) dt \right)^{1/q} \end{aligned} \quad (10)$$

provided the integrals on the right-hand side do exist.

**Proof.**

$$\begin{aligned}
 & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dx dy \\
 &= \int_0^\infty \int_0^\infty \frac{f(x)y^{(-1-\lambda)\frac{1}{p}}}{x^{(-1-\lambda)\frac{1}{q}} |x-y|^{(1-\lambda)/p} (\max\{e^{x/y}, e^{y/x}\})^{1/p}} \\
 &\quad \times \frac{g(y)x^{(-1-\lambda)\frac{1}{q}}}{y^{(-1-\lambda)\frac{1}{p}} |x-y|^{(1-\lambda)/q} (\max\{e^{x/y}, e^{y/x}\})^{1/q}} dx dy \\
 &\leq \left( \int_0^\infty \int_0^\infty \frac{f^p(x)y^{-1-\lambda}}{x^{(-1-\lambda)\frac{p}{q}} |x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dx dy \right)^{1/p} \\
 &\quad \times \left( \int_0^\infty \int_0^\infty \frac{g^q(y)x^{-1-\lambda}}{y^{(-1-\lambda)\frac{q}{p}} |x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dx dy \right)^{1/q} \\
 &= M^{1/p} N^{1/q}.
 \end{aligned}$$

We first consider

$$\begin{aligned}
 M &= \int_0^\infty \int_0^\infty \frac{f^p(x)y^{-1-\lambda}}{x^{(-1-\lambda)\frac{p}{q}} |x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dx dy \\
 &= \int_0^\infty x^{(1+\lambda)(p-1)} f^p(x) \left( \int_0^\infty \frac{y^{-1-\lambda}}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dy \right) dx.
 \end{aligned}$$

Now,

$$\int_0^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dy = M_1 + M_2,$$

where

$$M_1 = \int_0^x \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dy, \quad x > 0,$$

and

$$M_2 = \int_x^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dy, \quad x > 0.$$

Then

$$\begin{aligned}
 M_1 &= \int_0^x \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dy = \int_0^x \frac{y^{-\lambda-1}}{(x-y)^{1-\lambda} e^{x/y}} dy \\
 &= x^{-1} \int_0^1 \frac{u^{-\lambda-1} e^{-1/u}}{(1-u)^{1-\lambda}} du, \quad (y/x = u) \\
 &= \frac{\Gamma(\lambda)}{e} x^{-1},
 \end{aligned}$$

and

$$\begin{aligned}
 M_2 &= \int_x^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dy = \int_x^\infty \frac{y^{-\lambda-1}}{(y-x)^{1-\lambda} e^{y/x}} dy \\
 &\leq x^{-\lambda-1} \int_x^\infty \frac{1}{(y-x)^{1-\lambda} e^{y/x}} dy \\
 &= x^{-1} \int_1^\infty \frac{e^{-v}}{(v-1)^{1-\lambda}} dv, \quad y/x = v \\
 &= \frac{\Gamma(\lambda)}{e} x^{-1}.
 \end{aligned}$$

Therefore,

$$\int_0^\infty \frac{y^{-\lambda-1}}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dy \leq 2 \frac{\Gamma(\lambda)}{e} x^{-1},$$

and hence

$$M \leq 2 \frac{\Gamma(\lambda)}{e} \int_0^\infty x^{(1+\lambda)(p-1)-1} f^p(x) dx.$$

Similarly,

$$N \leq 2 \frac{\Gamma(\lambda)}{e} \int_0^\infty y^{(1+\lambda)(q-1)-1} g^q(y) dy.$$

Collecting the above estimations, we have

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{e^{x/y}, e^{y/x}\}} dx dy \\
 &\leq 2 \frac{\Gamma(\lambda)}{e} \left( \int_0^\infty t^{(1-\lambda)(p-1)-1} f^p(t) dt \right)^{1/p} \left( \int_0^\infty t^{(1-\lambda)(q-1)-1} g^q(t) dt \right)^{1/q}. \quad (11)
 \end{aligned}$$

□

**Theorem 4.** Assume that  $f, g \geq 0$ ,  $\lambda > 0$ ,  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dx dy \\ & \leq 2B(\lambda, \lambda) \left( \int_0^\infty t^\lambda f^p(t) dt \right)^{1/p} \left( \int_0^\infty t^\lambda g^q(t) dt \right)^{1/q}, \end{aligned} \quad (12)$$

provided the integrals on the right-hand side do exist.

**Proof.**

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dx dy \\ & = \int_0^\infty \int_0^\infty \frac{f(x)}{|x-y|^{\frac{1-\lambda}{p}} \left( \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\} \right)^{1/p}} \\ & \quad \times \frac{g(y)}{|x-y|^{\frac{1-\lambda}{q}} \left( \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\} \right)^{1/q}} dx dy \\ & \leq \left( \int_0^\infty \int_0^\infty \frac{f^p(x)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dx dy \right)^{1/p} \\ & \quad \times \left( \int_0^\infty \int_0^\infty \frac{g^q(y)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dx dy \right)^{1/q} \\ & = A^{1/p} C^{1/q}. \end{aligned}$$

First we estimate

$$\begin{aligned} A & = \int_0^\infty \int_0^\infty \frac{f^p(x)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dx dy \\ & = \int_0^\infty f^p(x) \left( \int_0^\infty \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy \right) dx. \end{aligned}$$

Now,

$$\int_0^\infty \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy = A_1 + A_2,$$

where

$$\begin{aligned}
 A_1 &= \int_0^x \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy \\
 &= \int_0^x \frac{(\frac{y}{x})^{2\lambda}}{(x-y)^{1-\lambda}} dy \\
 &= x^\lambda \int_0^x \frac{(\frac{y}{x})^{\lambda+1} (\frac{y}{x})^{\lambda-1} \frac{1}{x}}{(1-\frac{y}{x})^{1-\lambda}} dy \\
 &\leq x^\lambda \int_0^x \frac{(\frac{y}{x})^{\lambda-1} \frac{1}{x}}{(1-\frac{y}{x})^{1-\lambda}} dy \\
 &= x^\lambda \int_0^1 \frac{u^{\lambda-1}}{(1-u)^{1-\lambda}} du \\
 &= B(\lambda, \lambda) x^\lambda,
 \end{aligned}$$

and

$$\begin{aligned}
 A_2 &= \int_x^\infty \frac{1}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dy \\
 &= \int_x^\infty \frac{(\frac{y}{x})^{-2\lambda}}{(y-x)^{1-\lambda}} dy \\
 &= x^\lambda \int_x^\infty \frac{(\frac{y}{x})^{-2\lambda} \frac{1}{x}}{(\frac{y}{x}-1)^{1-\lambda}} dy \\
 &= x^\lambda \int_1^\infty \frac{u^{-2\lambda}}{(u-1)^{1-\lambda}} du \\
 &= B(\lambda, \lambda) x^\lambda.
 \end{aligned}$$

Therefore, we get

$$A \leq 2B(\lambda, \lambda) \int_0^\infty x^\lambda f^p(x) dx.$$

Similarly,

$$C \leq 2B(\lambda, \lambda) \int_0^\infty y^\lambda g^q(y) dy.$$

Collecting these estimates, we obtain

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^{1-\lambda} \max\{(\frac{x}{y})^{2\lambda}, (\frac{y}{x})^{2\lambda}\}} dx dy \\ & \leq 2B(\lambda, \lambda) \left( \int_0^\infty x^\lambda f^p(x) dx \right)^{1/p} \left( \int_0^\infty y^\lambda g^q(y) dy \right)^{1/q}. \end{aligned}$$

□

**Theorem 5.** Assume that  $f, g \geq 0$ ,  $0 < \lambda < 2$ ,  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{\frac{\lambda}{2}-1}} dx dy \\ & \leq B(\frac{\lambda}{2}, \frac{\lambda}{2}) \left( \int_0^\infty t^{1-\lambda} f^p(t) dt \right)^{1/p} \left( \int_0^\infty t^{1-\lambda} g^q(t) dt \right)^{1/q} \end{aligned} \quad (13)$$

provided the integrals on the right-hand side do exist.

**Proof.**

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{\frac{\lambda}{2}-1}} dx dy &= \int_0^\infty \int_0^\infty \frac{f(x)}{(x+y)^{\frac{\lambda}{p}} (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)\frac{1}{p}}} \\ &\quad \times \frac{g(y)}{(x+y)^{\frac{\lambda}{q}} (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)\frac{1}{q}}} dx dy \\ &\leq \left( \int_0^\infty \int_0^\infty \frac{f^p(x)}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dx dy \right)^{1/p} \\ &\quad \times \left( \int_0^\infty \int_0^\infty \frac{g^q(y)}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dx dy \right)^{1/q} \\ &= S^{1/p} T^{1/q}. \end{aligned}$$

We have

$$\begin{aligned} S &= \int_0^\infty \int_0^\infty \frac{f^p(x)}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dx dy \\ &= \int_0^\infty f^p(x) \left( \int_0^\infty \frac{1}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dy \right) dx \\ &= \int_0^\infty f^p(x) (S_1 + S_2) dx, \quad x > 0, \end{aligned}$$

where

$$\begin{aligned}
 S_1 &= \int_0^x \frac{1}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dy \\
 &= \int_0^x \frac{1}{(x+y)^\lambda (\frac{y}{x})^{\frac{\lambda}{2}-1}} dy \\
 &= x^{1-\lambda} \int_0^x \frac{(\frac{y}{x})^{2-\lambda} (\frac{y}{x})^{\frac{\lambda}{2}-1} \frac{1}{x}}{(1+\frac{y}{x})^\lambda} dy \\
 &\leq x^{1-\lambda} \int_0^x \frac{(\frac{y}{x})^{\frac{\lambda}{2}-1} \frac{1}{x}}{(1+\frac{y}{x})^\lambda} dy \\
 &= x^{1-\lambda} \int_0^1 \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^\lambda} du \\
 &= \frac{1}{2} B(\frac{\lambda}{2}, \frac{\lambda}{2}) x^{1-\lambda},
 \end{aligned}$$

and

$$\begin{aligned}
 S_2 &= \int_x^\infty \frac{1}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dy \\
 &= \int_x^\infty \frac{1}{(x+y)^\lambda (\frac{x}{y})^{\frac{\lambda}{2}-1}} dy \\
 &= x^{1-\lambda} \int_x^\infty \frac{(\frac{y}{x})^{\frac{\lambda}{2}-1} \frac{1}{x}}{(1+\frac{y}{x})^\lambda} dy \\
 &= \frac{1}{2} B(\frac{\lambda}{2}, \frac{\lambda}{2}) x^{1-\lambda}.
 \end{aligned}$$

Therefore, we have

$$\int_0^\infty \frac{1}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{(\frac{\lambda}{2}-1)}} dy \leq B(\frac{\lambda}{2}, \frac{\lambda}{2}) x^{1-\lambda},$$

which implies

$$S \leq B(\frac{\lambda}{2}, \frac{\lambda}{2}) \int_0^\infty x^{1-\lambda} f^p(x) dx.$$

Similarly,

$$T \leq B(\frac{\lambda}{2}, \frac{\lambda}{2}) \int_0^\infty y^{1-\lambda} g^q(y) dy.$$



Summarizing the above we obtain

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda (\min\{\frac{x}{y}, \frac{y}{x}\})^{\frac{\lambda}{2}-1}} dx dy \leq B(\frac{\lambda}{2}, \frac{\lambda}{2}) \left( \int_0^\infty t^{1-\lambda} f^p(t) dt \right)^{1/p} \\ \times \left( \int_0^\infty t^{1-\lambda} g^q(t) dt \right)^{1/q}.$$

□

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